Math 245B Lecture 4 Notes

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1 Separation Axioms and Urysohn's Lemma

1.1 Second countability and separability

Proposition 1.1. *Every second countable topological space is separable. In metric spaces, the converse is true.*

Proof. Let E be a countable base for $\mathcal{T}$. Pick one point $x_U \in U$ for all $U \in \mathcal{E}$. Now $\{x_U : U \in E\}$ is dense. Let $y \in X$, let V be a neighborhood of y. Now $V = \bigcup_{U \in \mathcal{E}'} U$ for some $\mathcal{E}' \subseteq \mathcal{E}$. If $U \in \mathcal{E}'$, then $u_U \in V$, so $V \cap \{x_U\} \neq \varnothing$.

Let (X, ρ) be a separable metric space, and let $A \subseteq X$ be countable and dense. Let $\mathcal{E} = \{B_r(x) : x \in A, r > 0, r \in \mathbb{Q}\}$. Check that this is a base:

1. $\bigcup_{x \in A} B_1(x) = X$ by the density of A.

2. Let $x, y \in A$, $r, s \in \mathbb{Q} \cap (0, \infty)$. Let $z \in B_r(x) \cap B_s(y)$. Pick $\delta > 0$ with $\delta \in \mathbb{Q}$ such that $B_{2\delta}(z) \subseteq B_r(x) \cap B_s(y)$. Let $w \in A$ be such that $\rho(z, w) < \delta$. Now $B_\delta(w) \ni z$, and $B_\delta(w) \subseteq B_{2\delta}(z) \subseteq B_r(x) \cap B_s(y)$. $\square$

The reverse implication is not true in general, but you have to deal with a complicated set theory construction.

1.2 Separation axioms

Definition 1.1. A topological space $(X, \mathcal{T})$ has property

1. T_0: For all $x, y \in X$ with $x \neq y$, there exists $U \in \mathcal{T}$ such that $|U \cap \{x, y\}| = 1$.

2. T_1: For all $x, y \in X$ with $x \neq y$, there exists $U \in \mathcal{T}$ such that $U \cap \{x, y\} = \{x\}$.

3. T_2 (**Hausdorff property**): If $x \neq y \in X$, there exist $U \neq V \in \mathcal{T}$ such that $U \cap T = \varnothing$, $x \in U$, and $y \in V$.

4. T_3 (**regular**): T_1 and whenever $x \in X$ and $A \subseteq X$ is closed, there exist $U, V \in \mathcal{T}$ such that $U \cap V = \varnothing$, $x \in U$, and $A \subseteq V$.

5. T_4 (**normal**): T_1 and whenever $A, B \subseteq X$ are closed and disjoint, there exist $U, V \in \mathcal{T}$ open such that $U \cap V = \varnothing$, $A \subseteq U$, and $B \subseteq V$.

Proposition 1.2. *$(X, \mathcal{T})$ is T_1 if and only if singletons are closed sets.*

Proof. ($\implies$): Let $\{x\} \in X$. If $y \in X \setminus \{x\}$, then by T_1, there exists $U)y \in \mathcal{T}$ such that $y\ inU_y$, $x \notin U_y$. Now $X \setminus \{x\} = \bigcup_y U_y \in \mathcal{T}$.

($\impliedby$): If $x \neq y \in X$, then $X \setminus \{x\}$ is open and contains $y$but not x. □

Corollary 1.1. $T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0$.

Lemma 1.1. *Any metric space $(X, \mathcal{T})$ is T_4.*

Proof. Assume $A, B \neq \varnothing$. $A \subseteq \{x : \rho(x, A) < \rho(x, B)\}$, and $B \subseteq \{x : \rho(x, A > \rho(x, B)\}$, where $\rho(x, A) = \inf_{y \in A} \rho(x, y)$. The function $x \mapsto \rho(x, A)$ is continuous, so these are open sets. □

1.3 Urysohn's lemma

Lemma 1.2 (Urysohn). *Let $(X, \mathcal{T})$ be T_4 and let $A, B \subseteq X$ be disjoint and closed. Then there exists $f \in C(X, [0, 1])$ such that $f|_A = 0$ and $f|_B = 1$.*

Remark 1.1. In metric spaces, we can just use the function

$$f(x) = \frac{\rho(x, A)}{\rho(x, A) + \rho(x, B)}.$$

The converse is true, as well; it is much easier to prove.

The idea is that if you had f, you could construct level sets (like when $f = 1/3$). We try to reconstruct f using its level sets.

Lemma 1.3. *Let $\Delta = \{k/2^n : n \geq 1, 0 < k < 2^n\}$. Then there exists $\{U_r : r \in \Delta\} \subseteq \mathcal{T}$ such that*

1. *$A \subseteq U_r \subseteq B^c$ for all r.*
2. *If $r < s \in \Delta$, then $\overline{U}_r \subseteq U_s$.*

Proof. We want to find $U_{1/2}$ such that $A \subseteq U_{1/2}$ and $\overline{U}_{1/2} \subseteq B^c$. By T_4, there exist $U \supseteq A$ and $V \supseteq B$ such that $U \cap V = \varnothing$; i.e. $U \subseteq V^c \subseteq B^c$, so $\overline{U} \subseteq B^c$. Now let $U_{1/2} := U$.

Suppose we have U_r for $r = k/2^n$ $n = 1, \ldots, m-1$. Consider U_s, where $s = d/2^n$. Let $r_1 = (\ell - 1)/2^n$,and $r_2 = (\ell + 1)/2^n$. Repeat the previous construction with the closed sets $\overline{U}_r$ and $U_{r_2}^c$. This gives us U_s. □

We can now prove Urysohn's lemma.

Proof. Let $\{U_r : r \in \Delta\}$ be given by the lemma. Define

$$f(x) := \begin{cases} \inf\{r \in \Delta : x \in U_r\} & \exists r \in \Delta \text{ s.t. } x \in U_n \\ 1 & x \notin \bigcup_{r \in \Delta} U_r \end{cases}$$

Suppose $x \in X$, $0 < f(x) < 1$. Let $\varepsilon > 0$. Choose $r_1 < r_2 \in \Delta \cap (f(x) - \varepsilon, f(x))$, $s \in \Delta \cap (f(x), f(x) + \varepsilon)$. Now $x \in U_{r_2}^c \supseteq (\overline{U}_{r_1})^c$, but $x \in U_s$. Now $U_s \cap (\overline{U}_{r_1})^c$ us a neighborhood V of x and $f[V] \subseteq (f(x) - \varepsilon, f(x) + \varepsilon)$. $\square$

Here is the final (but not very useful) separation axiom.

Definition 1.2. A topological space $(X, \mathcal{T})$ is $T_{3\,1/2}$ if for all $x \in X$ and $A \subseteq X$ closed with $x \notin A$, there exists $f \in C(X, [0, 1])$ such that $f(x) = 0$ and $f|_A = 1$.

This is a weakened version of the condition Urysohn's lemma which is weaker than T_4 and stronger than T_3.

Next time, we will use Urysohn's lemma to prove the following result.

Theorem 1.1 (Tietze's extension theorem)**.** *Let $(X, \mathcal{T})$ be T_4, let $A \subseteq X$ be closed, and let $f \in C(A, [a, b])$. Then there exists $F \in C(X, [a, b])$ such that $F|_A = f$. The same holds if $C(X, [a, b])$ is replaced with $C(X, K)$, where $K = \mathbb{R}$ or $\mathbb{C}$.*